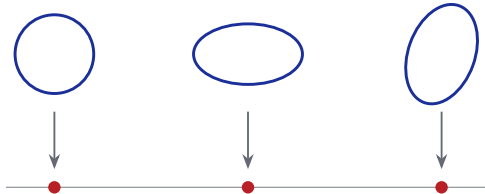


What is a moduli space?

An invitation to algebraic geometry



Master's colloqium, Leiden University

A space of geometric objects

We will explore examples, rather than give a fully formal definition.

The guiding idea

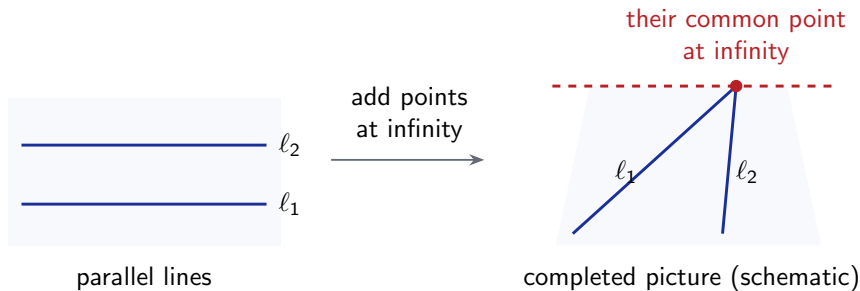
The **points of a space** represent **geometric objects**, and moving in the space describes how the objects vary.

Today: examples in **algebraic geometry**.

Moduli also appear in differential geometry, number theory, mathematical physics...

1. Projective space: adding points at infinity

Two distinct lines in the plane meet at one point. Except...

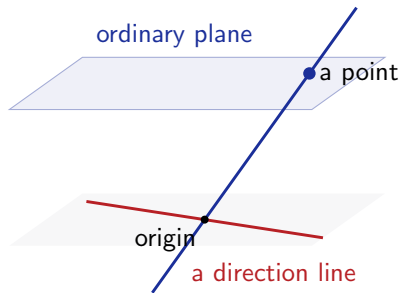


The projective plane

The completed plane is denoted $\mathbb{P}^2(\mathbb{R})$. Any two distinct projective lines meet exactly once.

The projective plane as a space of lines

Place an ordinary plane above the origin in $\mathbb{R}^3$.



A line through the origin either

- meets the plane in one point, or
- is parallel to it and represents a point at infinity.

Our first moduli space

$\mathbb{P}^2(\mathbb{R})$ parametrizes the lines through the origin in $\mathbb{R}^3$.

Projective space and homogeneous coordinates

Let k be a field.

$$\mathbb{P}^n(k) = \{\text{one-dimensional subspaces of } k^{n+1}\} = \frac{k^{n+1} \setminus \{0\}}{k^\times}.$$

A point has homogeneous coordinates

$$[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n], \quad \lambda \neq 0.$$

- Projective lines correspond to two-dimensional vector subspaces $W \subset k^{n+1}$.

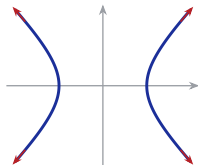
Polynomial equations in projective space

A polynomial $f(x_0, \dots, x_n)$ is **homogeneous of degree d** if

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n).$$

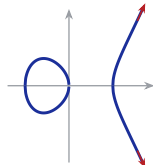
Thus $f = 0$ defines a well-defined subset

$$V(f) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : f(x_0, \dots, x_n) = 0\}.$$



Conic

$$x_0^2 - x_1^2 = x_2^2$$



Cubic

$$x_1^2 x_2 = x_0^3 - x_0 x_2^2$$

Affine chart $x_2 = 1$

Algebraic varieties

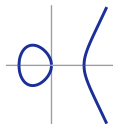
Projective algebraic geometry studies common zero loci of **homogeneous polynomials**:

$$X = V(f_1) \cap \cdots \cap V(f_r) \subseteq \mathbb{P}^n.$$

- A **curve** has dimension one.
- A **surface** has dimension two.



elliptic curve
complex points



plane cubic
genus 1, real chart

The next question

Can a collection of algebraic objects itself form an **algebraic space of parameters**?

2. What should a moduli space remember?

Choose a collection of geometric objects $\mathfrak{X}$.

An informal guiding principle

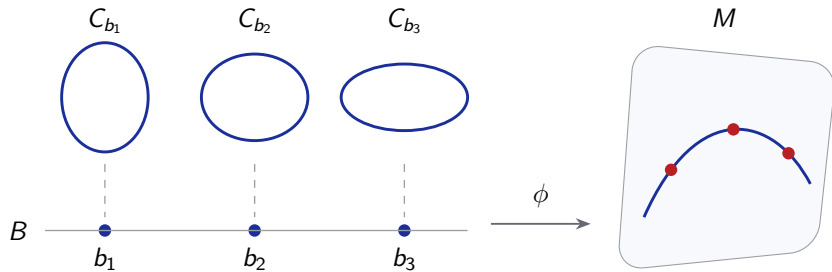
points of $M \longleftrightarrow$ objects in $\mathfrak{X}$ up to equivalence,

maps $B \longrightarrow M \longleftrightarrow$ families of objects over B .

- The first line gives a **classification**.
- The second line records **variation in families**.

Families become maps

A family $\mathcal{C} \rightarrow B$ assigns a curve C_b to every $b \in B$.



$$\phi : B \longrightarrow M, \quad b \longmapsto [C_b].$$

The Grassmannian

The Grassmannian parametrizes higher-dimensional vector subspaces:

$$\mathrm{Gr}(r, n+1) = \{W \subseteq k^{n+1} : \dim W = r\}.$$

In particular, $\mathrm{Gr}(2, 4)$ parametrizes lines in $\mathbb{P}^3$.

The Plücker embedding

Choose a basis u, v of W and put $p_{ij} = u_i v_j - u_j v_i$.

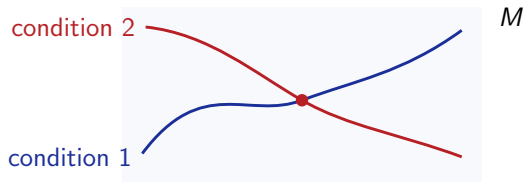
$$\mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}^5, \quad W \longmapsto [p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}].$$

Its image is the quadric hypersurface

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

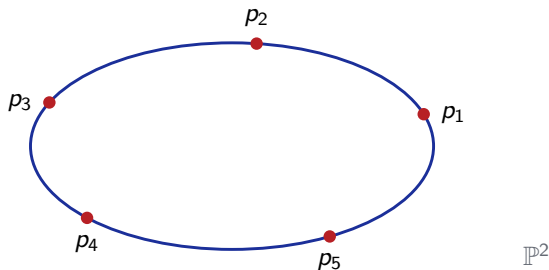
3. Counting with a moduli space

- A condition on objects in $\mathfrak{X}$ becomes a locus in M .
- Several conditions give an intersection of loci.
- A finite intersection gives the objects we want to count.



How many conics pass through five points?

Take five distinct points $p_1, \dots, p_5 \in \mathbb{P}^2(\mathbb{C})$, with **no three collinear**.



We will answer the question using **linear algebra on a moduli space**.

Conics form a projective space

Every plane conic has an equation

$$f_a = a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + a_{02}x_0x_2 + a_{12}x_1x_2 + a_{22}x_2^2.$$

Multiplying all coefficients by the same nonzero scalar changes nothing.

The parameter space of plane conics

$$[a_{00} : a_{01} : a_{11} : a_{02} : a_{12} : a_{22}] \in \mathbb{P}^5.$$



smooth conic



two lines



double line

We include **singular conics**.

A point condition is a hyperplane

Fix $p = [p_0 : p_1 : p_2] \in \mathbb{P}^2$.

The conics through p form the locus

$$H_p = \{[a] \in \mathbb{P}^5 : f_a(p) = 0\}.$$

Example: $p = [1 : 1 : 1]$

$$H_p = V(a_{00} + a_{01} + a_{11} + a_{02} + a_{12} + a_{22}) \subset \mathbb{P}^5.$$

a point in the plane $\longmapsto$ a **hyperplane** in the conic space.

Five conditions give one conic

The desired conics correspond to

$$H_{p_1} \cap \cdots \cap H_{p_5} \subset \mathbb{P}^5.$$

A weak form of Bézout's theorem

Any n hyperplanes in $\mathbb{P}_k^n$ have a common point.

Exercise. Prove this statement using linear algebra.

A harder exercise

If no three of $p_1, \dots, p_5$ are collinear, prove that the intersection is exactly **one point**, representing a smooth conic.

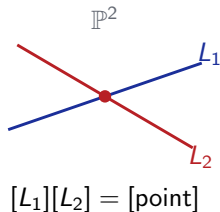
Intersection theory: counting with cohomology

Cohomology assigns a **graded ring** $H^*(X, \mathbb{Z})$ to a topological space.

For a smooth complex projective variety X , a subvariety Y gives a class

$$\text{codim}_{\mathbb{C}} Y = r \implies [Y] \in H^{2r}(X, \mathbb{Z}).$$

- **Multiplication** records intersections.
- **Integration** of a top-degree class gives an intersection number.



Conics: H is the hyperplane class of $\mathbb{P}^5$

$$\#\{\text{conics through the five points}\} = \int_{\mathbb{P}^5} H^5 = 1.$$

A different count: lines meeting four lines

How many lines meet **four general fixed lines** in $\mathbb{P}^3(\mathbb{C})$?

- The parameter space is the **Plücker quadric** $Q = \text{Gr}(2, 4) \subset \mathbb{P}^5$.
- **Exercise.** Show that meeting a fixed line gives a **hyperplane section** of Q .

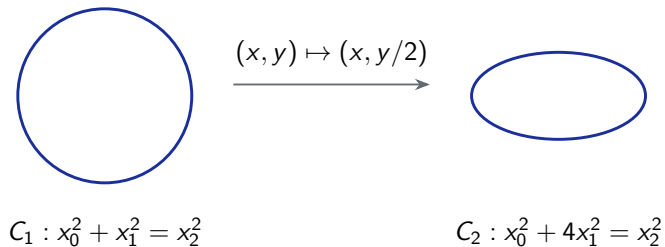
Write H for the hyperplane class in $\mathbb{P}^5$, and $h = H|_Q$. Since Q has degree two, $[Q] = 2H$.

The cohomology calculation

$$\int_Q h^4 = \int_{\mathbb{P}^5} H^4[Q] = 2 \int_{\mathbb{P}^5} H^5 = 2.$$

4. Different equations, the same object

Two conics can be different subsets of $\mathbb{P}^2$ and still be **isomorphic as abstract curves**.



$$C_1 \neq C_2 \text{ in the plane,} \quad C_1 \cong C_2 \cong \mathbb{P}^1 \text{ over } \mathbb{C}.$$

$\mathbb{P}^5$ classifies **embedded conics**.

Genus-one curves and the j -invariant

We now classify curves **without** fixing a projective embedding.

For a smooth cubic over $\mathbb{C}$:

$$j : V(y^2 - x^3 - ax - b) \longmapsto 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

$$j(C_1) = j(C_2) \iff C_1 \cong C_2.$$

Classification by the j -invariant

$$\underbrace{\mathcal{M}_1}_{\text{moduli of genus-one curves}} \xrightarrow{j} \underbrace{\mathbb{C}}_{j\text{-values}}$$

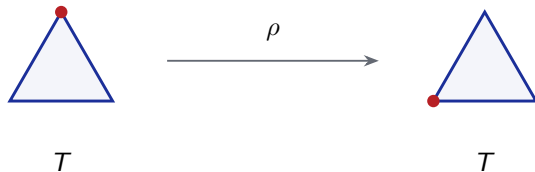
j is **not an isomorphism!**

Triangles and their automorphisms

$M_{\triangle}$ = moduli space of triangles, up to similarity.

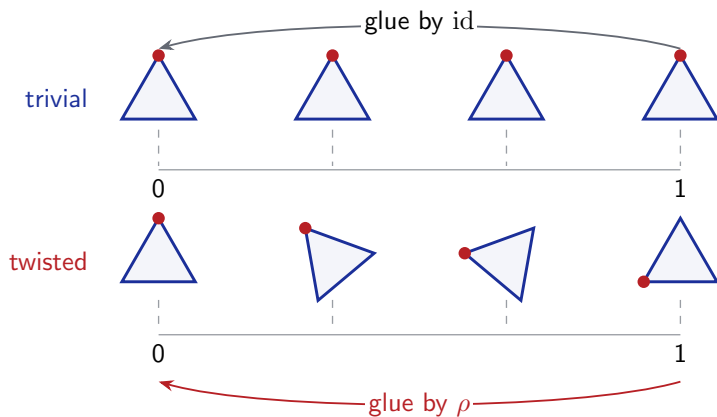
The equilateral triangle T has the **automorphism**

$$\rho : T \longrightarrow T, \quad \rho = \text{rotation through } 120^\circ, \quad \rho^3 = \text{id}.$$



Two families of triangles over S^1

Cut the circle at one point: $S^1 = [0, 1]/(0 \sim 1)$.

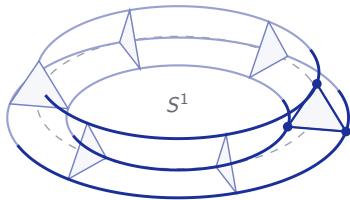


$$E_0 = S^1 \times T, \quad E_1 = ([0, 1] \times T) / ((1, x) \sim (0, \rho x)).$$

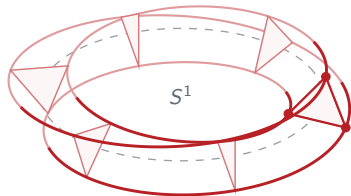
Constant maps, different families

Every triangle in either family is isomorphic to T . Both maps are constant:

$$\phi_0 = \phi_1 : S^1 \longrightarrow M_\Delta, \quad t \longmapsto [T].$$



Trivial family
Three separate vertex circles.



Twisted family
One vertex circle winds three times.

The families are not isomorphic !

Stacks

A **stack** = a space that also remembers the **automorphisms of its objects**.

space of isomorphism classes



stack



- A map $B \rightarrow \mathcal{M}$ describes a **family over B** , including its gluing.
- Isomorphisms between families are part of the data.

Algebraic moduli problems lead to **algebraic stacks**.

Genus-one curves and the j -invariant

We classify curves **without** fixing a projective embedding.

For smooth genus-one curves over $\mathbb{C}$,

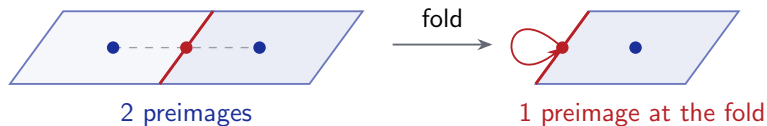
$$j(C_1) = j(C_2) \iff C_1 \cong C_2.$$

The j -map

$$\underbrace{\mathcal{M}_1}_{\text{moduli stack}} \xrightarrow{j} \underbrace{\mathbb{C}}_{\text{coarse moduli space}}$$

- The **j -line** records geometric isomorphism classes.
- The **stack** retains families and their automorphisms.

Counting on stacks



Count with automorphisms

$$\text{Automorphism group } G \implies \frac{1}{|G|}.$$

Stacky counts are “better”:

Orbit counting theorem: $G \curvearrowright X$,

$$\sum_{[x] \in X/G} \frac{1}{|\text{Stab}_G(x)|} = \frac{|X|}{|G|}.$$

Exercise

Show that the number of finite sets is e .

5. Modern moduli theory: curves

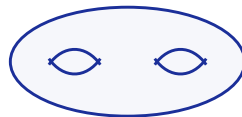
$$\mathcal{M}_g = \{\text{smooth connected projective curves of genus } g\}$$



$g = 0$



$g = 1$



$g = 2$

Topological pictures of complex curves

$\mathcal{M}_g$ is an algebraic stack of dimension $3g - 3$.

Counting curves over finite fields

$$\#\mathcal{M}_g(\mathbb{F}_p) := \sum_{\substack{C \text{ curve over } \mathbb{F}_p \\ g(C)=g}} \frac{1}{|\text{Aut}(C)|}.$$

g	$\#\mathcal{M}_g(\mathbb{F}_p)$	nonzero $H^k(\mathcal{M}_g, \mathbb{Q})$
1	p	H^0
2	p^3	H^0
3	$p^6 + p^5 + 1$	H^0, H^2, H^6

Each coefficient 1 matches a one-dimensional cohomology group.

Theorem (Bergström–Canning–Petersen–Schmitt, 2026)

$$\#\mathcal{M}_6(\mathbb{F}_p) = p^{15} + p^{14} + 2p^{13} + p^{12} - p^{10} + p^3 - 1.$$

Reading: moduli spaces and intersection theory

- K. Behrend, *Introduction to algebraic stacks*
Introductory examples of stacks and automorphisms.
- W. Fulton, *Intersection Theory*
Chow groups and algebraic intersection products.
- R. Pandharipande, *Reflections on moduli space* (2023)
A collection of geometers' perspectives on moduli.

Reading: algebraic geometry

- R. Hartshorne, *Algebraic Geometry*
Chapters II–III: schemes and cohomology. Chapter IV: curves.
- R. Vakil, *The Rising Sea*
A longer route into algebraic geometry, with examples and exercises.
- N. Nitsure, *Construction of Hilbert and Quot schemes*
In *FGA Explained*. A route to Hilbert schemes.

Questions?

Thank you.